Revisiting Solow's Decomposition of Economic and Productivity Growth

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Abstract

By relaxing the two assumptions of constant returns to scale and perfect competition in the product market used by Solow (1957), this paper identifies a new decomposition of economic and productivity growth. The sources of economic growth are; adjusted economies of scales effect, weighted sum of input growth, and technical progress. The sources of productivity growth are; adjusted economies of scale effect and technical progress. The weight used for the input growth is the cost share of each input.

Keywords: Solow; Growth decomposition; Total factor productivity; Returns to scale

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1. Introduction

The classical Solow (1957) article argues that the two components of economic growth are technical progress and input growth. The two assumptions Solow used are constant returns to scale and perfect competition in the product market. This paper relaxes these assumptions and derives new decomposition for both output growth and productivity growth.

2. Solow's Decomposition

Solow (1957) used the following simple Cobb-Douglas function with output *Y* and two inputs, labor (*L*) and capital (*K*), to demonstrate the decomposition of the economic growth at time *t*.

$$Y_t = A_t L_t^{\alpha} K_t^{\beta}, \tag{1}$$

where A_t measures the cumulative shift of the production function and α and β are parameters. Taking logarithm transformation and differentiation with respect to time *t*,

$$\dot{Y}_t = \dot{A}_t + \alpha \dot{L}_t + \beta \dot{K}_t, \tag{2}$$

where a dot over a variable represents the percentage change of the variable. For example, $\dot{Y}_t = \frac{\partial Y_t / \partial t}{Y_t}$. $\dot{A}_t$ represents technical progress. A constant returns to scale applies when $\alpha + \beta = 1$. We denote output per unit of labor as $y = \frac{Y}{L}$ and capital per unit of labor as $k = \frac{K}{L}$. Then

$$\dot{y}_t = \dot{A}_t + \beta \dot{k}_t. \tag{3}$$

The two major components for economic growth are technical progress and weighted input growth. The weight β is the capital share if factors are paid by their marginal products and the product market is perfectly competitive. Rearranging the terms, Solow (1957) considers the following estimation of the technical progress:

$$\dot{A}_t = \dot{y}_t - \beta \dot{k}_t . \tag{4}$$

This has led to the so-called "Solow residual." Given a measure of capital share, the technical progress can be estimated with the data on the growth in output, capital, and labor. The empirical application of the decomposition shown in Equations (3) and (4) is called the *growth accounting* approach, which has been standardized in growth textbooks and applied in different analysis (Romer 2001; Mankiw *et al*. 1992).

3. Perfect Competition

The Cobb-Douglas production function can be generalized into

$$Y_t = F(X_{1t}, X_{2t}, \cdots, X_{nt}, t), \tag{5}$$

where X_{it} are inputs. The inclusion of *t* in *F* allows the production function to shift over time, due probably to technical progress. Taking logarithms and differentiating Equation (5) with respect to time yields:

$$\dot{Y}_t = \sum_i \frac{\partial F}{\partial X_{it}} \frac{X_{it}}{F} \dot{X}_{it} + \frac{\partial F / \partial t}{F} . \tag{6}$$

The last term represents the technical progress. Let $\dot{A}_t = \frac{\partial F / \partial t}{F}$. Then

$$\dot{Y}_t = \sum_i \frac{\partial F}{\partial X_{it}} \frac{X_{it}}{F} \dot{X}_{it} + \dot{A}_t . \tag{7}$$

Intuitively, $\frac{\partial F}{\partial X_{it}}\frac{X_{it}}{F}$ can be considered as a weight. With the two given assumptions, Solow (1957) argued that this weight is equal to the cost share.

Denote w_{it} as the nominal price of input X_{it} and P as the price of output. Under perfect competition in both product and factors markets with given P and w, the first order condition of maximizing profit $PF - \sum w_{it} X_{it}$ with respect to X_{it} gives $\frac{\partial F}{\partial X_{it}} = \frac{w_{it}}{P}$. Equation (7) then becomes

$$\dot{Y}_t = \sum_i \frac{w_{it} X_{it}}{PF} \dot{X}_{it} + \dot{A}_t . \tag{8}$$

The assumption that the product market is perfectively competitive implies that P is equal to average cost, PF is equal to total cost (C), and $\frac{w_{it} X_{it}}{PF} = \frac{w_{it} X_{it}}{C_t}$. Denote the cost share for input X_{it} as $s_{it} = \frac{w_{it} X_{it}}{C_t}$, Equation (8) becomes

$$\dot{Y}_t = \sum_i s_{it} \dot{X}_{it} + \dot{A}_t . \tag{9}$$

Equation (9) shows that output growth can be decomposed into two components: the weighted sum of input growth and technical progress. The weight is the cost share of each input. Equation (9) can be simplified into Equation (3) if F is a Cobb-Douglas production function with constant returns to scale and two inputs.

In short, the key assumptions in Solow's (1957) decomposition are constant returns to scale and perfect competition in the product market. The different forms of the decomposition based on different assumptions have been analyzed. For example, Azzam *et al*. (2004) consider the productivity growth under imperfect competition.

4. Imperfect Competition and Returns to Scale

The *weight* ($\frac{\partial F}{\partial X_{it}}\frac{X_{it}}{F}$) in Equation (7) is actually the output elasticity with respect to input X_{it}. Denote $\eta_{it} = \frac{\partial F}{\partial X_{it}}\frac{X_{it}}{F}$. Let $\eta_t = \sum_i \eta_{it}$ (the sum of the elasticity to each input). We can show that η_t is a measure of economies of scale. Suppose the changes in all inputs have the same scale, $\Delta X_{it} = aX_{it}$. Consider the changes in output ΔF by taking the total derivative of $F(X_1, X_2, \cdots, X_n, t)$ and substituting $\Delta X_{it} = aX_{it}$ into ΔF, we have

$$\Delta F = \sum_i \frac{\partial F}{\partial X_{it}}\Delta X_{it} + \frac{\partial F}{\partial t}\Delta t = F\sum_i \frac{\partial F}{\partial X_{it}}\frac{aX_{it}}{F} + F\dot{A}_t = Fa\sum_i \eta_{it} + F\dot{A}_t$$
$$= aF\eta_t + F\dot{A}_t. \qquad (10)$$

The production shows increasing (constant, decreasing) returns to scale when $\eta_t > 1$ (= 1, < 1).

We now consider the following cost minimization problem under perfect competition in the factors markets, but not necessary in the product market.

$$\min_{X_{it}} C_t = \sum_i w_{it}X_{it} \text{ subject to } Y_t = F(X_{1t}, X_{2t}, \cdots, X_{nt}, t). \qquad (11)$$

Write the objective function and the constraint in the Lagrangian form as

$$L(X_{it}, \lambda) = \sum_i w_{it}X_{it} + \lambda(Y_t - F), \qquad (12)$$

where λ is the Lagrange multiplier. The first-order condition for minimization is

$$w_{it} = \lambda\frac{\partial F}{\partial X_{it}}. \qquad (13)$$

The Lagrange multiplier has the following property:

$$\lambda = \frac{\partial C_t}{\partial Y_t}, \tag{14}$$

where *C* is the minimized cost in the minimization problem. Substitute (14) into (13),

$$w_{it} = \frac{\partial C_t}{\partial Y_t} \times \frac{\partial F}{\partial X_{it}}. \tag{15}$$

This implies that input price is equal to the marginal revenue of product, assuming marginal cost is equal to marginal revenue. Multiplying both side by X_{it} and divided by the total cost (*C*), we get the following cost share equation:

$$\frac{w_{it} X_{it}}{C_t} = \frac{\partial C}{\partial Y_t} \frac{\partial F}{\partial X_{it}} \frac{X_{it}}{C_t} = \frac{\partial C}{\partial Y_t} \frac{Y_t}{C_t} \frac{\partial F}{\partial X_{it}} \frac{X_{it}}{F}. \tag{16}$$

Denote $\theta_t = \frac{\partial C_t}{\partial Y_t} \frac{Y_t}{C_t}$ as the cost elasticity with respect to output. Then

$$\frac{w_{it} X_{it}}{C_t} = \theta_t \eta_{it}. \tag{17}$$

Taking the sum for all inputs gives $1 = \theta_t \sum_i \eta_{it}$ and $1 = \theta_t \eta_t$. Then

$$\eta_t = \theta_t^{-1}. \tag{18}$$

This implies that output elasticity to input is the inverse of the cost elasticity to output (Hanoch, 1975). When production is increasing (decreasing, constant) returns to scale,

the cost elasticity to output is less than (greater than, equal to) one. Substituting Equation (18) into Equation (17) and rearranging the terms, the weight is

$$\frac{\partial F}{\partial X_{it}}\frac{X_{it}}{F} = \eta_t \frac{w_{it}X_{it}}{C_t}. \tag{19}$$

Substituting (19) into the output growth Equation (7) gives

$$\dot{Y}_t = \eta_t \sum_i \frac{w_{it}X_{it}}{C_t}\dot{X}_{it} + \dot{A}_t, \tag{20}$$

By subtracting and adding $\sum_i \frac{w_{it}X_{it}}{C_t}\dot{X}_{it}$ and rearranging terms to consider the unit economies of scale, Equation (20) becomes

$$\dot{Y}_t = (\eta_t - 1)\sum_i \frac{w_{it}X_{it}}{C_t}\dot{X}_{it} + \sum_i \frac{w_{it}X_{it}}{C_t}\dot{X}_{it} + \dot{A}_t. \tag{21}$$

Output growth can now be decomposed into three components: adjusted economies of scale effect, weighted sum of input growth, and technical progress. For the first component, the economies of scale effect $\eta_t - 1$ is adjusted by the growth of aggregate input $\sum \frac{w_{it}X_{it}}{C_t}\dot{X}_t$. Denote $MC_t = \frac{\partial C_t}{\partial Y_t}$ as the marginal cost and $AC_t = \frac{Y_t}{C_t}$ as the average cost. Then $\theta_t = \frac{MC_t}{AC_t}$ and the term $\eta_t - 1$ becomes

$$\eta_t - 1 = \theta_t^{-1} - 1 = \frac{AC_t}{MC_t} - 1 = \frac{AC_t - MC_t}{MC_t}. \tag{22}$$

The economies of scale effect is determined by the difference between AC_t and MC_t. This difference can be considered as a markup effect when the market is not perfectly

competitive. For the second component, the weight for the growth of aggregate input is the cost share of each input.

The decomposition in Equation (21) has two advantages over Solow's (1957) decomposition. First, the assumption of constant returns to scale is no longer required since η_t need not equal to one. When production is constant returns to scale, $\eta_t = 1$, Equation (21) is reduced to Equation (9). This implies that the perfect competition in the product market is not required to derive Equation (9); only the constant returns to scale is sufficient. Second, the assumption of perfect competition in the product market is not required for the growth decomposition formula. Equation (21) can be used for an imperfectly competitive industry or economy with either increasing, constant, or decreasing returns to scale.

Equations (17) and (18) gives $\frac{w_{it} X_{it}}{C_t} = \frac{\eta_{it}}{\eta_t}$. Then Equation (21) becomes

$$\dot{Y}_t = (\eta_t - 1)\sum_i \frac{\eta_{it}}{\eta_t} \dot{X}_{it} + \sum_i \frac{\eta_{it}}{\eta_t} \dot{X}_{it} + \dot{A}_t . \tag{23}$$

Equation (23) shows the decomposition without cost information (*w*). As long as the parameters of the production function can be estimated, Equation (23) can be used for the empirical estimation of the sources of output growth.

The above decomposition of economic growth can be extended to the decomposition of productivity growth. The productivity of a production function with single output (Y_t) and single input (X_t) at time *t* is $\frac{Y_t}{X_t}$. For a production function with multiple inputs, the total factor productivity (TFP) can be defined as

$$TFP_t = \frac{Y_t}{\Phi_t}, \tag{24}$$

where Φ is the aggregate input. Taking logarithm and differentiation with respect to time, the TFP growth is

$$\dot{TFP}_t = \dot{Y}_t - \dot{\Phi}_t . \tag{25}$$

A commonly used measure of input growth is the Divisia index (Jorgenson and Griliches 1967).

$$\dot{\Phi}_t = \sum_i \frac{w_{it} X_{it}}{C_t} \dot{X}_{it} . \tag{26}$$

Substituting Equations (21) and (26) into (25), the TFP growth is

$$\dot{TFP}_t = (\eta_t - 1) \sum_i \frac{w_{it} X_{it}}{C_t} \dot{X}_{it} + \dot{A}_t . \tag{27}$$

Or,

$$\dot{TFP}_t = (\eta_t - 1) \sum_i \frac{\eta_{it}}{\eta_t} \dot{X}_{it} + \dot{A}_t . \tag{28}$$

The TFP growth now has two components: adjusted economies of scale effect and technical progress. The decomposition in Equation (28) is similar to Equation (8.2.6) in Kumbhakar and Lovell (2000, pp. 284). When production is constant returns to scale, $\eta_t = 1$, the decomposition is reduced to $\dot{TFP} = \dot{A}$ as in Solow (1957).

We can compare Equation (27) with the results from other studies. Substituting Equation (18) into Equation (27) gives

$$\dot{TFP} = (\theta^{-1} - 1) \sum_i \frac{w_{it} X_{it}}{C_t} \dot{X}_{it} + \dot{A}_t . \tag{29}$$

This equation is similar to the Equation (17) in Denny *et al*. (1982, p. 193) and the Equation (3) in Bauer (1990). These authors consider the decomposition based on the cost function approach. Since the cost elasticity to output is the inverse of the output

elasticity to input and the cost information is not required, the use of cost function approach is not necessary for the decomposition analysis.

5. Conclusions

This paper shows the decomposition of economic growth and productivity growth without Solow's (1957) assumptions of constant returns to scale and perfect competition in the product market. When these two assumptions are relaxed, the decomposition of economic growth and productivity growth include an additional term: the adjusted economies of scale effect.

In contrast to the constant returns to scale in the classical growth models (Solow, 1956; Mankiw *et al*., 1992), the endogenous growth model (Romer 1990; Aghion and Howitt, 1992) considers economic growth with increasing returns to scale due to the presence of R&D. Our decomposition provides a new growth accounting approach for increasing returns to scale. This approach is particularly convenient for modeling the growth of the regulated industries with increasing returns to scale (Cowing and Stevenson, 1981), and can be used for imperfect competition. For studies using cross-section data with the use of dummy variables for different industries or provinces, this approach can be used to differentiate industrial or provincial characteristics. For studies using time series data, this approach can be used to check if the returns to scale change over time. Our results also show that the production function approach is sufficient for the decomposition of the TFP growth.

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